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SOLUTION OF SELF-DUALITY EQUATION IN QUANTUM-GROUP GAUGE THEORY AND QUANTUM HARMONICS

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Abstract

We discuss the gauge theory for quantum group $SU_q(2) \times U(1)$ on the quantum Euclidean space. This theory contains three physical gauge fields and one $U(1)-$gauge field with a zero field strength. We construct the quantum-group self-duality equation (QGSDE) in terms of differential forms and with the help of the field-strength decomposition. A deformed analog of the BPST-instanton solution is obtained. We consider a harmonic (twistor) interpretation of QGSDE in terms of $SU_q(2)/U(1)$ quantum harmonics. The quantum harmonic gauge equations are formulated in the framework of a left-covariant 3D differential calculus on the quantum group $SU_q(2)$.

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An attractive idea of quantum deformations for the gauge theories has been considered in the framework of different approaches [1 - 6]. Formally one can discuss independent deformations of basic spaces and gauge groups and possible correlations between these deformations. We shall here consider a gauge theory with identical one-parameter deformations of the 4-dimensional Euclidean space and the gauge group $SU(2)$. A consistent formulation of the gauge theory for the semisimple quantum group $SU_q(N)$ is unknown to us, so we shall deal with the quantum group $U_q(2) = SU_q(2) \times U(1)$. It will be shown that the $U(1)$-gauge field can be treated as a field with a zero field strength.

Consider the standard relations between elements T^i_k of the quantum $U_q(2)$-matrix [8]

$$RT_1T_2 = T_1T_2R\ , \qquad R^2 = I + (q - q^{-1})R \tag{1}$$

where I is a unity matrix , R is a constant symmetric matrix with components $R^{ik}_{lm}(q)$ $(i, k, l, m = 1,\ 2)$ and q is a real deformation parameter.

It is convenient to use the following covariant representation for a deformed antisymmetric symbol

$$\varepsilon_{ik}(q) = \sqrt{q(ik)}\ \varepsilon_{ik} = -q(ik)\varepsilon_{ki}(q) \tag{2}$$

$$q(12) = [q(21)]^{-1} = q, \qquad q(11) = q(22) = 1 \tag{3}$$

$$\varepsilon_{ik}(q)\varepsilon^{kl}(q) = \delta^l_i\ ,$$

where ε_{ik} is an ordinary antisymmetric symbol $(\varepsilon_{ik} = \varepsilon^{ki})$.

The R-matrix can be written in terms of projection operators $P^{(+)}$ and $P^{(-)}$[8]

$$R = qP^{(+)} - q^{-1}P^{(-)} = qI - (q + q^{-1})P^{(-)} \tag{4}$$

$$P^{(+)}\ +\ P^{(-)} = I\ , \tag{5}$$

$$(P^{(\pm)})^2 = P^{(\pm)}, \quad P^{(+)}P^{(-)} = 0$$

where matrix $P^{(-)}$ has the following components

$$[P^{(-)}]^{ik}_{lm} = -\frac{q}{1+q^2}\varepsilon^{ki}(q)\varepsilon_{ml}(q) \tag{6}$$

We shall use also covariant representation for the $SU_q(2)$-metric

$$\mathcal{D}^i_k(q) = -\varepsilon_{mk}(q)\varepsilon^{mi}(q) \tag{7}$$

The basic RTT-relations imply the simple equation

$$\varepsilon_{ml}(q)T^l_i\ T^m_k = \varepsilon_{ki}(q)\ D(T) \tag{8}$$

where $D(T)$ is the quantum determinant [8].

A covariant expression for the inverse quantum matrix $S(T) = T^{-1}$ contains inverse determinant

$$S^i_k = \varepsilon_{kl}(q)\ T^l_j\ \varepsilon^{ji}(q)\ D^{-1}(T) \tag{9}$$

We shall use the well-known equations for multiplication of the transposed matrices

$$T_i^l \mathcal{D}_l^m(q) S_m^k = \mathcal{D}_i^k(q) \tag{10}$$

The unitarity condition for the matrix T can be formulated with the help of involution [8]

$$T_k^i \rightarrow \overline{T_k^i} = S_i^k \tag{11}$$

Let us consider the bicovariant differential calculus on the $U_q(2)$ group [9 - 12]

$$\begin{gathered} T_1 dT_2 = R dT_1 T_2 R \\ D(T) dT = q^2 dT D(T) \\ [d, T] = dT, \qquad \{d, dT\} = 0 \end{gathered} \tag{12}$$

Note that the condition $D(T) = 1$ is inconsistent in the framework of this calculus. Consider the relations for the right-invariant differential forms $\omega = dTS$[12]

$$\omega R \omega + R \omega R \omega R = 0 \tag{13}$$

$$T_1 \omega_2 = R \omega_1 R T_1 \tag{14}$$

The quantum trace ξ of the form ω plays an important role in this calculus

$$\begin{gathered} \xi(T) = \mathcal{D}_i^k(q) \omega_k^i(T) \neq 0, \qquad \xi^2 = 0, \qquad d\xi = 0 \\ dT = \omega T = (q^2 \lambda)^{-1} [T, \xi], \qquad q dD(T) = \xi D(T) \\ d\omega = \omega^2 = -(q^2\lambda)^{-1} \{\xi, \omega\} \end{gathered} \tag{15}$$

The bicovariant calculus makes the basis for consistent formulation of quantum-group gauge theory in the framework of noncommutative algebra of differential complexes [5-7]. Consider formally the quantum group gauge matrix $T_b^a(x)$ defined on some basic space. Suppose that Eqs(12-15) satisfy locally for each "point" x. Then one can try to construct the $U_q(2)$-connection 1-form $A_b^a(x)$ which obeys the simplest commutation relation

$$A\ R\ A + R\ A\ R\ A\ R\ =\ 0 \tag{16}$$

Note that the general relation for A contains a nontrivial right-hand side [7].

Coaction of the gauge quantum group $U_q(2)$ has the following standard form:

$$\begin{gathered} A \rightarrow T(x)\ A\ S(T)\ +\ dT(x)\ S(T) = T\ A\ S\ +\ \omega(T) \\ \alpha = \mathrm{Tr}_q A \rightarrow \alpha + \xi(T) \end{gathered} \tag{17}$$

The restriction $\alpha = 0$ is inconsistent with (16), but we can use the gauge-covariant relations

$$\alpha^2 = 0\ , \qquad \mathrm{Tr}_q A^2 = 0 \tag{18}$$

It should be stressed that we can choose the zero field-strength condition $d\alpha = 0$ for the $U(1)$-gauge field [1]. This constraint is gauge invariant and consistent with (16). The

[1]This condition is consistent also for the case of $GL_q(N)$ group

deformed pure gauge field α can be decoupled from the set of physical fields in the limit $q = 1$. We shall further consider the $U_q(2)$-gauge theory with three "physical" gauge fields and one "zero-mode" $U(1)$ field.

The curvature 2-form is q-traceless for this model

$$F = dA - A^2, \qquad \mathrm{Tr}_q F = 0 \tag{19}$$

Quantum deformations of Minkowski and Euclidean 4-dimensional spaces have been considered in Refs [13-15]. We shall treat the coordinates x^i_α of q-deformed Euclidean space $E_q(4)$ as generators of a noncommutative algebra covariant under the coaction of the quantum group $G_q(4) = SU^L_q(2) \times SU^R_q(2)$

$$R^{ik}_{lm} x^l_\alpha x^m_\beta = x^i_\gamma x^k_\rho R^{\gamma\rho}_{\alpha\beta} \tag{20}$$

where we use two identical copies of R-matrices (4) for left and right $SU_q(2)$-indices.

Coactions of the commuting left and right $SU_q(2)$ groups conserve (20)

$$x^i_\alpha \rightarrow l^i_k x^k_\beta r^\beta_\alpha \tag{21}$$

The q-deformed central Euclidean interval τ can be constructed by analogy with the quantum determinant

$$\tau = \mid x \mid^2 = -\frac{q}{1+q^2} \varepsilon^{\beta\alpha}(q) \varepsilon_{ki}(q) x^i_\alpha x^k_\beta \tag{22}$$

We do not consider the quantum group structure on $E_q(4)$. It is convenient to use the following $E_q(4)$ involution

$$\overline{x^i_\alpha} = \varepsilon_{ik}(q) x^k_\beta \varepsilon^{\beta\alpha}(q) = \tau S^\alpha_i(x) \tag{23}$$
$$\overline{\tau} = \tau \ , \quad \overline{\overline{x^i_\alpha}} = x^i_\alpha$$

We shall use an analog of the bicovariant $U_q(2)$-calculus (12-15) for studying differential complexes on $E_q(4)$. Consider the right-invariant
1-forms

$$\omega(x)^i_k = dx^i_\alpha S^\alpha_k(x) \tag{24}$$

Basic 2-forms on $E_q(4)$ can be decomposed with the help of $P^(\pm)$
operators (6)

$$dx^i_\alpha dx^k_\beta = \frac{q}{1+q^2}[\varepsilon_{\beta\alpha}(q) d^2x^{ik} \ + \ \varepsilon^{ki}(q) d^2 x_{\alpha\beta}] \tag{25}$$

By analogy with the classical case we can treat these two parts as self-dual and anti-self-dual 2-forms under the action of a duality operator $*$. It is convenient to rewrite this decomposition in terms of the right-invariant self-dual and anti-self-dual forms

$$P^{(-)} dx_1 dx_2 P^{(+)} = q P^{(-)}(q^3\omega^2 + \omega\xi) P^{(+)} x_1 x_2 P^{(+)} \tag{26}$$
$$P^{(+)} dx_1 dx_2 P^{(-)} = -(1/q) P^{(+)}(\omega R \omega) P^{(-)} \tau = (q^{-1}\omega\xi - \omega^2) P^{(-)} \tau \tag{27}$$

Let us introduce the simple ansatz for quantum $U_q(2)$ anti-self-dual gauge fields

$$A^a_b = dx^i_\alpha A^{\alpha a}_{ib}(x) = \omega^a_b(x) f(\tau) \ , \tag{28}$$
$$A^{\alpha a}_{ib}(x) = \delta^a_i S^\alpha_b(x) f(\tau) \ ,$$

where $f(\tau)$ is a function of the q-interval (22). Note that this ansatz is a partial case of more general construction of differential complex on $GL_q(2)$ [5,7]. Addition of the term $\xi(x)g(\tau)$ results in a relation for the connection A more complicated than (16) .

Consider the q-traceless curvature form for the connection (29)

$$F = \omega^2 f(\tau)[1 - f(q^2\tau)] + (q^2\lambda)^{-1}\omega\xi[f(\tau) - f(q^2\tau)] \tag{29}$$

The anti-self-duality equation $*F = -F$ for our ansatz is equivalent to the nonlinear finite-difference equation

$$f(\tau) - f(q^2\tau) = (1 - q^2)f(\tau)[1 - f(q^2\tau)] \tag{30}$$

This equation has a simple solution analogous to the classical BPST-solution

$$f(\tau) = \frac{\tau}{c + \tau} \ , \tag{31}$$

where c is an arbitrary constant. Note that our solution for connection A contains parameter q only through definitions of $\omega(x)$ and τ , however, the corresponding curvature has a more explicit q-dependence.

The curvature form can be written in terms of the field strength

$$F = dx^i_\alpha dx^k_\beta F^{\beta\alpha}_{ki}(x) = d^2x_{\alpha\beta}F^{\beta\alpha} + d^2x^{ik}F_{ki} \ ,$$

where Eq(25) is used.

The QGSD-equation for the field strength has the following form

$$F^{\beta\alpha}_{ki} = \varepsilon_{ki}(q)F^{\beta\alpha} \tag{32}$$

It is interesting to discuss the q-deformation of the harmonic (or twistor) formalism for QGSDE. The q-deformed harmonics can be considered as elements of $SU_q(2)$ matrix u^i_a, ($i = 1,\ 2$, $a = +,\ -$). We shall treat these matrix elements as coordinates of the noncommutative coset space $SU_q(2)/U(1)$ by analogy with the classical harmonic formalism for a self-duality equation[16].

Consider $SU_q(2) \times U(1)$ cotransformations of q-harmonics

$$u^i_\pm \rightarrow l^i_k u^k_\pm \exp(\pm i\alpha) \ , \tag{33}$$

where α is $U(1)$parameter and l is a matrix of left $SU_q(2)$ acting on $E_q(4)$.

The q-harmonics satisfy the following relations:

$$\begin{gathered} Ru_1u_2 = u_1u_2R \ , \qquad qu_1x_2 = Rx_1u_2 \ , \\ \varepsilon_{ki}(q)u^i_- u^k_+ = \sqrt{q} \ , \end{gathered} \tag{34}$$

It is convenient to use the 3-dimensional left-invariant differential calculus on $SU_q(2)$ [9, 17] for the harmonic formalism. Consider the q-traceless left-invariant 1-forms $\theta = S(u)du$ and introduce the notations:

$$\theta_0 = \theta^+_+ = -q^{-2}\theta^-_-, \qquad \theta_{(+2)} = \theta^-_+, \qquad \theta_{(-2)} = \theta^+_- \tag{35}$$

We shall below write the left-invariant relations between θ and u which allow us to define the operator of harmonic external derivative on $SU_q(2)$

$$d_u = \delta_0 + \delta + \bar{\delta} = \theta_0 D_0 + \theta_{(-2)} D_{(+2)} + \theta_{(+2)} D_{(-2)} \ , \tag{36}$$

where δ_0 , δ and $\bar{\delta}$ are invariant operators satisfying the ordinary Leibniz rules and the following relations

$$\begin{gathered} \delta_0^2 = \delta^2 = \bar{\delta}^2 = 0 \\ \{\delta_0, \delta\} + \left\{\delta_0, \bar{\delta}\right\} + \left\{\delta, \bar{\delta}\right\} = 0 \end{gathered} \tag{37}$$

The left-invariant differential operators $D_0, \ D_{(\pm 2)}$ are the basis of a q-deformed Lie algebra equivalent to the universal enveloping algebra $\mathbf{U}_q[SU(2)]$ [17].

Let us define an invariant decomposition of the Maurer-Cartan equations for $SU_q(2)/U(1)$

$$\begin{aligned} d_u\theta_0 &= 2\delta\theta_0 = 2\bar{\delta}\theta_0 = -\theta_{(-2)}\theta_{(+2)} \\ d_u\theta_{(+2)} &= \delta_0\theta_{(+2)} = q^2(1+q^2)\theta_0\theta_{(+2)} \\ d_u\theta_{(-2)} &= \delta_0\theta_{(-2)} = q^2(1+q^2)\theta_{(-2)}\theta_0 \end{aligned} \tag{38}$$

Define also δ_0 , δ and $\bar{\delta}$ operators on the quantum harmonics

$$\delta_0 u^i_+ = u^i_+\theta_0 = q^2\theta_0 u^i_+ \ , \qquad \delta u^i_+ = 0 \tag{39}$$
$$\bar{\delta} u^i_+ = u^i_-\theta_{(+2)} = q^{-1}\theta_{(+2)}u^i_-$$

$$\delta_0 u^i_- = -q^2 u^i_-\theta_0 = -\theta_0 u^i_- \ , \qquad \bar{\delta} u^i_- = 0 \tag{40}$$
$$\delta u^i_- = u^i_+\theta_{(-2)} = q\theta_{(-2)}u^i_+$$

Global functions on a quantum 2-sphere can be defined via the invariant condition

$$\delta_0 f(u) = \theta_0 D_0 f(u) = 0$$

Consider the harmonic decomposition of the Euclidean coordinates and derivatives

$$x_{(b)\alpha} = -q\varepsilon_{ik}(q)u^k_b x^i_\alpha \ , \qquad \partial^\alpha_a = u^i_a \partial^\alpha_i \tag{41}$$
$$\partial^\alpha_a x_{(b)\beta} = \delta^\alpha_\beta \varepsilon_{ba}(q)$$

where $a, \ b = +, \ -$.

One can use the asymmetric decomposition of the operator d_x on $E_q(4)$

$$d_x = dx^i_\alpha \partial^\alpha_i = \bar{d} + (d_x - \bar{d}) \tag{42}$$
$$\bar{d} \ \sim \ dx_{(-)\alpha}\partial^\alpha_+ \ , \qquad \bar{d}^2 = 0 \ , \qquad \left\{\bar{d}, \delta\right\} = 0$$

It should be remarked that the use of the symmetric decomposition results in a modification of analitycity condition for corresponding invariant operators.

An analyticity condition for the deformed harmonic space has the following form

$$\partial^\alpha_+ \Lambda(x_{(+)}, u) = 0 \iff \bar{d}\Lambda = 0$$

Multiplying QGSDE (32) by $u^i_+ u^k_+$ one can obtain the q-deformed integrability conditions in central basis (CB)(CB corresponds to u-independent gauge-group matrices $T(x)$) which are analogous to the classical self-dual integrability conditions [16 , 18].

Consider the decomposition of the $U_q(2)$-connection in CB corresponding to (42) and let $\bar{a} \sim dx_{(-)\alpha} A^\alpha_+(x)$ be a connection for $\bar{d}$. The quantum-group self-duality equation (32) is equivalent to the following zero-curvature equation

$$\bar{d}\bar{a} - \bar{a}^2 = 0 \tag{43}$$

This equation has the following harmonic solution

$$\bar{a} = \bar{d}hS(h) = \omega(h, \bar{d}h) \tag{44}$$

where $h(x,u)$ is a "bridge" $U_q(2)$-matrix function. The matrix elements of h and $\bar{d}h$ satisfy the relations analogous to Eqs(12-15). Additional harmonic relations are

$$\delta\bar{a} = 0\ , \quad \delta_0 h = 0 \tag{45}$$

The bridge solution possesses a nontrivial gauge freedom

$$h \ \rightarrow T(x)h\Lambda(x_{(+)}, u)\ , \quad \delta_0\Lambda = 0$$

where Λ is an analytical $U_q(2)$ gauge matrix.

The matrix h can be treated as a transition matrix from the central basis to the analytical basis (AB) where $\bar{d}$ has no connection. The characteristic feature of AB is a nontrivial harmonic connection V that is an invariant component of a new AB-basis A_{AB} in the algebra of $U_q(2)$ differential complexes

$$A_{AB} = S(h)Ah - S(h)dh = \tilde{A} - S(h)d_u h = \tilde{A} + V \tag{46}$$
$$V = -S(h)\delta h - S(h)\bar{\delta}h = v + \bar{v}$$

where the analytical connection $v = \theta_{(-2)}V_{(+2)}$ contains the analytical prepotential $V_{(+2)}$.

By analogy with the classical harmonic formalism [16] the prepotential $V_{(+2)}$ generates a general solution of QGSDE that can be obtained as a solution of the basic harmonic gauge equation

$$\delta h + hv = 0 \tag{47}$$

One can obtain explicit or perturbative solutions of this equation by using the noncommutative generalizations of classical harmonic expansions and harmonic Green functions [16 , 19]. It seems very interesting to study reductions of QGSDE to lower dimensions and to search a more general deformation scheme for the self-duality equation.

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